MTH 605 Midterm Solutions

1. (a) Let G be a topological group, and let H be a closed set in G. Show that if $H \trianglelefteq G$, then G/H is a topological group.

   (b) Show that $\mathbb{R}/\mathbb{Z}$ is a topological group. Is this space homeomorphic to any space known to you?

   Solution. (a) Since the group operation is continuous in $G \times G$, the left multiplication map $L_g : G \to G$, $x \mapsto g \cdot x$, is continuous. Furthermore, this map has an inverse is given by $L_{g^{-1}}$, which is also continuous. This shows that L_g is a homeomorphism. Similarly, the right multiplication map R_g is also a homeomorphism. Hence, if U is open in G, then for any subset A of G, UA is an open set, as $UA = \cup_{a \in A} Ua$. Therefore, the natural map $q : G \to G/H$ is an open and continuous map. Thus, $G \to G/H$ is a quotient map, and G/H is a group that is endowed with the quotient topology. Finally the fact that the map $G \times G \to G/H \times G/H$ is an open map and $gH \approx g^{-1}H$ would imply that G/H is a topological group. (Note that the the closedness of H was not required in the proof above.)

   (b) Since $(\mathbb{Z}, +) \trianglelefteq (\mathbb{R}, +)$ and $\mathbb{Z}$ is a closed subspace of $\mathbb{R}$ under the standard topology, $\mathbb{R}/\mathbb{Z}$ is an Hausdorff topological group. (Note that H is a closed subgroup of G iff G/H is Hausdorff. Try to prove this fact!) Under the equivalence induced by the quotient map $\mathbb{R} \to \mathbb{R}/\mathbb{Z}$ all intervals of the form $[n, n+1], n \in \mathbb{Z}$ will be identified. In particular, $\mathbb{R}/\mathbb{Z} \approx [0,1]/0 \sim 1 \approx S^1$.

2. The *Borsuk-Ulam Theorem* states that for every continuous map $f : S^n \to \mathbb{R}^n$, there exists $x \in S^n$ such that $f(-x) = f(x)$. Using the Borsuk-Ulam Theorem, show that if $S^2 = A_1 \cup A_2 \cup A_3$, where each A_i is a closed set, then one of the sets A_i must contain an antipodal pair of points $\{x, -x\}$. [Hint: Use the map $f_i(x) = \text{dist}(x, A_i)$.]

   Solution. Let $f_i(x) = \text{dist}(x, A_i)$ for $i = 1, 2$. Since A_i is compact, $x \in A_i$ iff $\text{dist}(x, A_i) = 0$. Then $f = f_1 \times f_2$ is a continuous map $S^2 \to \mathbb{R}^2$. By the Borsuk-Ulam Theorem, there exists $x \in S^2$ such that $f_i(x) = f_i(-x)$.

   If $f_1(x) = 0$, then $f_1(-x) = 0$, and so $\{x, -x\} \subset A_1$. Similarly, if $f_2(x) = 0$, then $f_2(-x) = 0$, so $\{x, -x\} \subset A_2$. Otherwise, $f_i(\pm x) > 0$, and we must have $\{x, -x\} \subset A_3$.

3. (a) Let C be a path component of X and let $x_0 \in C$ be a basepoint. Show that the inclusion map $C \hookrightarrow X$ induces an isomorphism of fundamental groups $\pi_1(C, x_0) \to \pi_1(X, x_0)$.

   (b) Let $p : \widetilde{X} \to X$ be a simply connected covering space. Let $A \subset X$ be path-connected and locally path-connected, and let $\widetilde{A}$ be a path component of $p^{-1}(A)$. Then show that $p|_{\widetilde{A}} : \widetilde{A} \to A$ is a covering space corresponding to the kernel of the homomorphism $\pi_1(A) \to \pi_1(X)$.

Solution. (a) If $f : I \to X$ is a loop at x_0, then since C is path connected, the image of f is in C. Therefore f may be written as $i \circ f'$, where $i : C \to X$ is inclusion and f' is the map f with restricted range C. Now $[f] = [i \circ f'] = i_*([f'])$ and so $i_* : \pi_1(C, x_0) \to \pi_1(X, x_0)$ is surjective.

If $i_*([f]) = 1$ in $\pi_1(X, x_0)$ then $i \circ f \simeq e_{x_0}(via\, H)$. Since $I \times I$ is path connected, $H(I \times I) \subset C$. Restricting the range of H gives a path homotopy H' such that $f \simeq e_{x_0}(via\, H')$ in C. Hence $[f] = [e_{x_0}]$ in $\pi_1(C, x_0)$ and i_* is injective.

(b) We know that $p|_{p^{-1}(A)} : p^{-1}(A) \to A$ is a covering space, and from (a), the restriction of a covering space to a path component is a also a covering space. Let $x_0 \in A$ and $y_0 \in \widetilde{A}$ be basepoints with $p(y_0) = x_0$.

Let $i : A \to X$ and $j : \widetilde{A} \to \widetilde{X}$ be the inclusion maps, and let $p' = p|_{\widetilde{A}}$. Then $p \circ j = i \circ p'$, and consequently, $p_* \circ j_* = i_* \circ p'_*$. Since $p_* \circ j_*$ factors through $\pi_1(\widetilde{X}, y_0)$, which is trivial, we have that $i \circ p'$ is trivial. Hence, $\text{Im}(p'_*) \subset \ker(i_*)$.

It remains to show that $\ker(i_*) \subset \text{Im}(p'_*)$. Let $[f] \in \ker(i_*)$, where f is a loop in A based at x_0. Let $\widetilde{f}$ be the unique lift of f to $\widetilde{A}$ with initial point y_0. Then $j \circ \widetilde{f}$ is a lift of $i \circ f$ to $\widetilde{X}$ with initial point y_0. Since $[i \circ f] = [e_{x_0}]$ in $\pi_1(X, x_0)$, this latter lift is in fact a loop in $\widetilde{X}$. Hence $\widetilde{f}$ is a loop in $\widetilde{A}$, and $[f] = [p' \circ \widetilde{f}]$ in $\pi_1(A, x_0)$, that is, $\ker(i_*) \subset \text{Im}(p'_*)$
.

4. (a) Let $r : X \to A$ be a retraction map. For $x_0 \in A$, what can you say about the homomorphism $j_* : \pi_1(A, x_0) \to \pi_1(X, x_0)$ induced by the inclusion $j : A \hookrightarrow X$?

   (b) Show that the fundamental group of the figure eight space (i.e. the union of two circles touching at a point) is infinite.

Solution. (a) Since $r \circ j = i_A$, we have that $r_* \circ j_* = (i_A)_*$, which is the identity isomorphism. Hence, j_* is injective.

(b) Let the two circles be given by (A, x) and (B, y), where x and y are chosen basepoints. Then the figure eight space in the quotient space $A \sqcup B / x \sim y$. (Note that such a quotient space is called a *wedge product* of A and B written as $X = A \vee B$.) Consider the map $r : X \to A$ defined by $r|_A = i_A$ and $r|_B = e_y$. By the Pasting Lemma R is continuous, and since $r|_A = i_A$, r is a retraction onto A. From (a), the $\pi_1(A, x) \hookrightarrow \pi_1(X, x)$. Since $\pi_1(A, x) \cong \mathbb{Z}$, $\pi_1(X, x)$ has to be infinite.

5. (a) Let $\{U_1, U_2\}$ be an open covering of a space X such that $U_1 \cap U_2$ is nonempty and path-connected, and each U_i is simply connected. Show that X is simply connected. [Hint: Consider the open covering $\{f^{-1}(U_i)\}$ of I, and use the Lebesque number lemma.]

   (b) Prove that S^n is simply connected for $n \geq 2$.

   Solution. (a) We prove the following generalization of (a): Let $\mathcal{U}$ be an open cover of a space X such that each set of $\mathcal{U}$ is simply connected and each intersection of two sets of $\mathcal{U}$ is path connected. Then X is simply connected.

   Proof. Let $a : I \to X$ be a loop in X at the base point x. By the Lebesgue covering lemma, there is a subdivision $a = a_1 + a_2 + \cdots + a_n$ of a such that each a_i lies in some set U_i of $\mathcal{U}$. We consider a_i as a map $I \to X$. Let b_0, b_{n+1} be constant paths at the point x. By the assumptions, we can for $1 < i \leq n$ choose a path $b_i : I \to X$ joining $a_i(0)$ to x and lying in $U_{i-1} \cap U_i$. The loop $-b_i + a_i + b_{i+1}$ lies in U_i and so is contractible to a constant map (via a homotopy that keeps the end points fixed), by assumption of simple connectivity of U_i. It follows that a is contractible to a point in X.

   (b) Let N and S denote the north and south poles of S^n respectively for $n \geq 2$. Then $S^n \setminus \{N\} \approx \mathbb{R}^n \approx S^n \setminus \{S\}$ via stereographic projections. Let $U_1 = S^n \setminus \{N\}$ and $U_2 = S^n \setminus \{S\}$, then $S^1 = U_1 \cup U_2$, where each U_i is simply connected, and $U_1 \cap U_2 = S^n \setminus \{N, S\}$ is path-connected. From (a), we have that S^n for $n \geq 2$ has to be simply-connected.

6. A subspace A is *deformation retract* of X, if there exists a retraction $r : X \to A$ homotopic to the identity map i_X on X.

(a) Show that if A is a deformation retract of X, then $\pi_1(X, x_0) \cong \pi_1(A, x_0)$ for any $x_0 \in A$. [Hint: Use the fact that if $f, g : (X, x_0) \to (Y, y_0)$ and $f \simeq g\,(via\, H)$ such that $H(x_0, t) = y_0, \forall\, t$, then $f_* = g_*$.]

(b) Show that S^n is a deformation retract of $R^{n+1} \setminus \{0\}$ for $n \geq 1$.

(c) Show that $\mathbb{R}^2$ cannot be homeomorphic to $\mathbb{R}^n$ for $n > 2$.

Solution. (a) Since $r : X \to A$ is a retraction, we know from 4(a) that $j_* : \pi_1(A, x_0) \hookrightarrow \pi_1(X, x_0)$. From the fact stated in the hint, we have that $(i_x)_* = j_* \circ r_*$. From this we infer that j_* is surjective, and hence j_* is an isomorphism.

(b) There is a natural retraction $r : R^{n+1} \setminus \{0\} \to S^n, x \mapsto x/\|x\|$. This retraction is homotopic to the identity map on $R^{n+1} \setminus \{0\}$ via $H(x, t) = (1 - t)x + tx/\|x\|$. Hence $R^{n+1} \setminus \{0\}$ deformation retracts onto S^n.

(c) Suppose we assume on contrary that $\mathbb{R}^2 \approx \mathbb{R}^n$ for $n > 2$. Then $R^2 \setminus \{0\} \approx R^{n+1} \setminus \{0\}$. From(a) and (b), this would imply that $\pi_1(S^1) \cong \pi_1(S^n)$, which is impossible, as $\pi_1(S^n)$ is simply-connected for $n > 2$ (from 5(b)).

7. **Bonus.** Let $\phi : \mathbb{R}^2 \to \mathbb{R}^2$ be the linear transformation $\phi(x, y) = (2x, y/2)$. Note that ϕ defines an equivalence $\sim$ on $X = \mathbb{R}^2 \setminus \{0\}$ defined by $(x_1, y_1) \sim (x_2, y_2)$ iff $(x_2, y_2) = \phi((x_1, y_1))$. Let $p : X \to X/\sim$ be the induced quotient map.

   (a) Show this p is a covering space.

   (b) Show the orbit space $X/\sim$ is non-Hausdorff, and describe how it is a union of four subspaces homeomorphic to $S^1 \times \mathbb{R}$, coming from the complementary components of the x-axis and the y-axis.

   (c) What is the fundamental group of $X/\sim$?

Solution. (a) First we need to show that every point has on open neighborhood that is disjoint from all of its translates under powers of ϕ. For the point (a, b) a product neighborhood of the form $(c, 2c) \times \mathbb{R}$ where $a/2 < c < a$ (if $a \neq 0$) or $\mathbb{R} \times (d, 2d)$, where $b/2 < d < b$ (if $b \neq 0$) will work. Hence, $p : X \to X/\sim$ is a covering space map.

(b) First, note that ϕ defines a group action of $\mathbb{Z}$ on X. This action preserves the subset $(0, \infty) \times \mathbb{R}$, and identifies each line $x = a$ homeomorphically to the line $x = 2a$. Hence the image of this set is homeomorphic to $S^1 \times \mathbb{R}$. The same is true of the subsets $(\infty, 0) \times \mathbb{R}$, $\mathbb{R} \times (0, \infty)$, and $\mathbb{R} \times (\infty, 0)$. Since these sets cover X, the quotient is a union of four copies of $S^1 \times \mathbb{R}$. Each line of the form $\{a\} \times \mathbb{R}$ in the annulus $((0, \infty) \times \mathbb{R})/\mathbb{Z}$ spirals around and limits onto two circles (the images of the two halves of the y-axis). Similarly, the circle $S^1 \times \{0\}$ is the limiting circle for lines in two of the other annuli.

(c) We know that S has fundamental group $\mathbb{Z}$ and the group of covering translations of this covering is also $\mathbb{Z}$. Hence $\pi_1(X/\mathbb{Z})$ maps onto $\mathbb{Z}$ with kernel isomorphic to $\mathbb{Z}$. It follows that $\pi_1(X/\mathbb{Z})$ is a semidirect product $\mathbb{Z} \rtimes \mathbb{Z}$. There are only two such groups, $\mathbb{Z} \times \mathbb{Z}$ and a non-abelian group (because there are only two automorphisms of $\mathbb{Z}$). These generators are given by loops in $X/\mathbb{Z}$ as follows: one is the image of the loop γ in X which generates $\pi_1(X, x_0)$; the other is the image of a path α in X joining the basepoint x_0 to $\phi(x_0)$. It is not difficult to map $I \times I$ into X so that its boundary maps to the path $\gamma * \alpha * \phi(\overline{\gamma}) * \overline{\alpha}$. This is possible because ϕ preserves the orientation of X. Then the boundary of the image of this square in $X/\mathbb{Z}$ represents the commutator of the generators, and so they commute. Hence, $\pi_1(X/\mathbb{Z}) \cong \mathbb{Z} \times \mathbb{Z}$.